Math 105 Chapter 5: Techniques of Integration

Uptil now we have be working with integrals an their properties, but as we have seen, computing integrals is not fun. It can be quite tedius and difficult working with reimann summs. In the last chapter we derived the fundamental theorem of calculus.

Theorem: (FTC II) If f is continuous on $[a,b]$ and F is an antiderivative of f on $[a,b]$ then

$$\int_a^b f(x)dx = F(b) - F(a)$$

So the above theorem says that if we have an antiderivative of f, then FTC tells us that we can evaluate an integral by plugging in 2 points! So we have turned the problem of finding an area into a problem of finding antiderivatives! We will spend the remainder of this section doing just that!

First note that we know the antiderivative of all the derivatives we computed in calc I. eg

$$\frac{d}{dx}\frac{x^{n+1}}{n+1} = x^n \implies \int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq 1$$

$$\frac{d}{dx}\log|x| = \frac{1}{x} \implies \int \frac{1}{x}dx = \log|x| + C$$

$$\frac{d}{dx}\sin x = \cos x \implies \int \cos x\, dx = \sin x + C$$

Just like in calculus 1, the derivatives of the elementary functions were the building blocks of the more difficult ones, by using chain and product rule.

We will develop analogous rules integrals.

Substitution

The analog of chain rule is substitution and $F' = f$ then

$$\frac{d}{dx}(F(g(x))) = f(g(x))\, g'(x)$$

$$\Rightarrow \quad F(g(x)) = \int f(g(x))\, g'(x)\, dx \quad (\bigstar)$$

If $u = g(x)$ then then $\frac{du}{dx} = g'(x) \Rightarrow du = g'(x)dx$. Plugging into $(\bigstar)$ we get

$$\boxed{\int f(g(x))\, g'(x)\, dx = \int f(u)\, du}$$

The above is called Substitution. Instead of explaining that equation, lets do some examples:

eg $\int (2x+100)^{50} dx$

Sol^n Let $u = 2x + 100$, $\frac{du}{dx} = 2 \Rightarrow dx = \frac{du}{2}$

So $\int (2x+100)^{50} dx = \int u^{50} \frac{du}{2}$

$$= \frac{1}{2} \int u^{50} du$$

$$= \frac{1}{2} \cdot \frac{1}{51} u^{51} + C$$

Thus $\int (2x+100)^{50}dx = \frac{1}{102}u^{51} + C$

$$= \frac{(2x+100)^{51}}{102} + C$$

↑ don't forget!

eg $\int x e^{x^2} dx$

Sol^{n} Let $u = x^2$, $\frac{du}{dx} = 2x \Rightarrow \frac{du}{2} = x\,dx$

$$\Rightarrow \int x e^{x^2} dx = \int e^u \frac{du}{2}$$

$$= \frac{1}{2}\int e^u du$$

$$= \frac{1}{2}e^u + C$$

$$= \frac{1}{2}e^{x^2} + C$$

eg $\int \sin^3 x \cos x\,dx$

$u = \sin x$, $du = \cos x\,dx$

$$\Rightarrow \int \sin^3 x \cos x\,dx = \int u^3 du$$

$$= \frac{1}{4}u^4 + C$$

$$= \frac{1}{4}\sin^4 x + C$$

Let take a break from examples to make a few remarks!

1. When do we use substitution?

Substitutions transform our integral into new (potentially) easier integrals.

2. If we make a substitution $u(x)$ then we want to integrate with respect to u, so we need to turn all the "x" in our integral into a u, including the du.

eg We want to make sure the derivative of u is next to the dx!

$$\text{eg } \int x e^{x^2} dx, \quad \text{we let } u = x^2, \; du = 2x\,dx$$

note $\frac{d}{dx}(x^2) = 2x$ is next to the dx (upto a constant multiple).

3. We can even use substitution to solve definite integrals!

$$\text{eg } \int_0^2 \frac{1}{x+3}\,dx$$

$u = x+3, \; du = dx$

$$\text{Now } \int_0^2 \frac{1}{x+3}\,dx \neq \int_0^2 \frac{1}{u}\,du$$

In the first integral the limits of integration were $x=0$ and $x=2$, so we need to determine what u is when $x=0, 2$.

$$x=0 \implies u = 0+3 = 3$$

$$x=2 \implies u = 2+3 = 5$$

So $\displaystyle\int_{x=0}^{x=2} \frac{1}{x+3}\,dx = \int_{u=3}^{u=5} \frac{1}{u}\,du$

$$= \Big[\log|u|\Big]_{u=3}^{u=5}$$

$$= \log 5 - \log 3$$

$$= \log\left(\tfrac{5}{3}\right)$$

and $\displaystyle\int \frac{1}{x+3}\,du = \log|x+3|$

MORE EXAMPLES!

eg $\displaystyle\int \tan x\,dx$

$= \displaystyle\int \frac{\sin x}{\cos x}\,dx$, $u = \cos x$, $du = -\sin x\,dx$

$= \displaystyle\int -\frac{du}{u}$

$= -\log|u|$

$= -\log|\cos x|$

$= \log|\sec x|$

eg $\int x\sqrt{x+1}\,dx$, $u = x+1$, $du = dx$

$x = u-1$

$= \int (u-1)\sqrt{u}\,du$

$= \int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du$

$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$

$= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$

eg $\int_0^2 x(x^2+4)^5\,dx$, $u = x^2+4$, $\frac{du}{2} = x\,dx$

$u(0)=4$, $u(2)=8$

$= \int_4^8 u^5 \frac{du}{2}$

$= \left[\frac{1}{12}u^6\right]_4^8$

$= \frac{8^6}{12} - \frac{4^6}{12}$

Now lets do an example where we need to make 2 substitutions.

eg. $\int \frac{\log(\tan x)}{\sin x \cos x}\,dx$.

The part of the integral that is giving us the trouble log of tan.
We can simplify by getting rid of the tan by a substitution

let $u = \tan x$, $du = \sec^2 x\, dx$
$dx = \cos^2 x\, du$

So $\int \frac{\log(\tan x)}{\sin x \cos x} dx$

$= \int \frac{\log u}{\sin x \cos x} \cos^2 x\, du$

$= \int \log u \cdot \frac{\cos x}{\sin x} du$

$= \int \log u \cdot \frac{1}{\tan x} du$

$= \int \frac{\log u}{u} du$, since $\tan x = u$

Now we use substitution again, to deal with the log.

$w = \log u$, $dw = \frac{1}{u} dw$

$$\int \frac{\log(\tan x)\, dx}{\sin x \cos x} = \int w\, dw$$
$$= \frac{1}{2} w^2 + C$$
$$= \frac{1}{2}(\log u)^2 + C$$
$$= \frac{1}{2}(\log(\tan x))^2 + C$$

Integration by Parts

Just like substitution was the "integral version" of chain rule, Integration by parts (IBP) is the "integral version" of product rule.

Recall: $$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$

$$\Rightarrow \underbrace{\int \frac{d}{dx}(uv)\,dx}_{uv} = \int v \underbrace{\frac{du}{dx}dx}_{du} + \int u \underbrace{\frac{dv}{dx}dx}_{dv}$$

$$\Rightarrow uv = \int v\,du + \int u\,dv$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

The last formula is called the integration by parts formula.

If we took definite integrals above we would get

$$\boxed{\int_a^b u\,dv = uv\Big]_a^b - \int_a^b v\,du}$$

IBP is useful when:

1. the integrand is a product of functions, u & dv.
2. u is easy to <u>differentiate</u>, dv is easy to integrate
3. $v\,du$ is easier to evaluate than $u\,dv$.

Lets do examples.

eg $\int x e^x dx$

Let $u = x$, $dv = e^x dx$

$\Rightarrow du = dx$, $v = e^x$

So $\int x e^x dx = uv - \int v du$, by IBP

$= xe^x - \int e^x dx$

$= xe^x - e^x + C$

Note if $u = e^x$, $dv = x\,dx$

$\Rightarrow du = e^x dx$, $v = \frac{x^2}{2}$

So $\int xe^x dx = \frac{x^2}{2} e^x - \int \frac{x^2}{2} e^x dx$

Note $\int \frac{x^2}{2} e^x dx$ is even worse than $\int xe^x dx$ because of the square.

So the choice of u, dv is very important.

For the next example we will see sometimes you need to use IBP twice.

eg $\int_0^{\pi} x^2 \sin x \, dx$

$u = x^2$, $dv = \sin x dx$

$du = 2x dx$, $v = -\cos x$

So $\int_0^{\pi} x^2 \sin x \, dx = -x^2 \cos x \Big|_0^{\pi} - \int_0^{\pi} 2x(-\cos x)\, dx$

$= -\pi^2 \cos \pi + \int_0^{\pi} 2x \cos x \, dx$

$= \pi^2 + 2\int_0^{\pi} x \cos x \, dx$

To find $\int_0^{\pi} x \cos x dx$, we use IBP again.

$u = x$, $dv = \cos x dx$

$du = dx$, $v = \sin x$

So $\int_0^{\pi} x \cos x \, dx = x \sin x \Big|_0^{\pi} - \int_0^{\pi} \sin x \, dx$

$= 0 + [\cos x]_0^{\pi}$

$= \cos \pi - \cos 0$

$= -2$

Thus $\int_0^{\pi} x^2 \sin x = \pi^2 + 2(-4)$

$= \pi^2 - 4$

eg $\int e^x \sin x \, dx$

Let $u = e^x$, $dv = \sin x dx$

$du = e^x dx$, $v = -\cos x$

So $\int e^x \sin x dx = -e^x \cos x + \int e^x \cos x \, dx$

Now note $\int e^x \cos x \, dx$ is basically our original integral but with $\cos x$ instead of $\sin x$. So lets apply IBP again, and see what we get

$u = e^x$ $\quad dv = \cos x \, dx$

$du = e^x dx$ $\quad v = \sin x$

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

Thus

$$\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx$$

$$\Rightarrow \quad 2\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x$$

$$\Rightarrow \quad \int e^x \sin x\,dx = \frac{e^x(\sin x - \cos x)}{2}$$

Sometimes IBP is useful even when there doesn't seem like there is a product.

eg $\int \log x\,dx$, $u = \log x$, $dv = dx$, $du = \frac{1}{x}dx$, $v = x$

$$= x\log x - \int \frac{1}{x}\cdot x\,dx$$

$$= x\log x - \int dx$$

$$= x\log x - x + C$$

<u>excercize</u>: Show $\int \arctan x\,dx = x\arctan x - \frac{1}{2}\log|x^2+1| + C$

$$\int \arcsin x\,dx = x\arcsin x + \sqrt{1-x^2} + C$$

eg $\int \sin x \cos x \log(\sin x)\, dx$

Let us make the substitution $u = \sin x$, since the $\log(\sin x)$ is particularly offensive to the eyes. Let $u = \sin x$ will simplify that.

so $u = \sin x$, $du = \cos x$

$$\Rightarrow \int \sin x \cos x \log(\sin x)\, dx$$

$$= \int u \log u \, du$$

Now lets apply integration by parts.

$$\Rightarrow \quad w = \log u \qquad dv = u\,du$$
$$dw = \frac{du}{u} \qquad v = \frac{u^2}{2}$$

$$\Rightarrow \int u \log u \, du = \frac{u^2}{2} \log u - \int \frac{u^2}{2} \cdot \frac{1}{u}\, du$$

$$= \frac{u^2}{2} \log u - \int \frac{u}{2}\, du$$

$$= \frac{u^2}{2} \log u - \frac{u^2}{4} + C$$

$$\Rightarrow \int \sin x \cos x \log(\sin x)\, dx = \frac{u^2}{2} \log u - \frac{u^2}{4} + C$$

$$= \frac{\sin^2 x}{2} \log(\sin x) - \frac{\sin^2 x}{4} + C$$

Odd / Even Functions

Definition: A function is even if for all x

$$f(-x) = f(x)$$

A function is odd if for all x

$$f(-x) = -f(x)$$

eg. If n is even then x^n is even since

$$(-x)^n = x^n$$

In n is odd then x^n is odd since

$$(-x)^n = -x^n$$

This is the motivation for the terms odd/even.

eg $\sin x$ is odd, $\cos x$ is even since

$$\sin(-x) = -\sin x, \quad \cos(-x) = \cos x$$

Properties of odd even functions:

1. odd + odd = odd   eg $x^5 + \sin x$

   even + even = even   eg $x^{10} + 2\cos x$

   odd + even = neither, if both functions are non-zero.

   eg. $f(x) = x^2 + \sin x$

   $f(-x) = (-x)^2 + \sin(-x) = x^2 - \sin x \neq f(x), -f(x)$

(2) even $\times$ even = even   eg $x^2 \sec x$

odd $\times$ odd = even   eg $x \sin x$

odd $\times$ even = odd   eg $x \cos x$, $\tan x = \frac{\sin x}{\cos x}$

(3) even (even) = even   eg $\cos(x^2)$

even (odd) = even   eg $(\sin x)^2$

odd (even) = even   eg $\sin(x^2)$

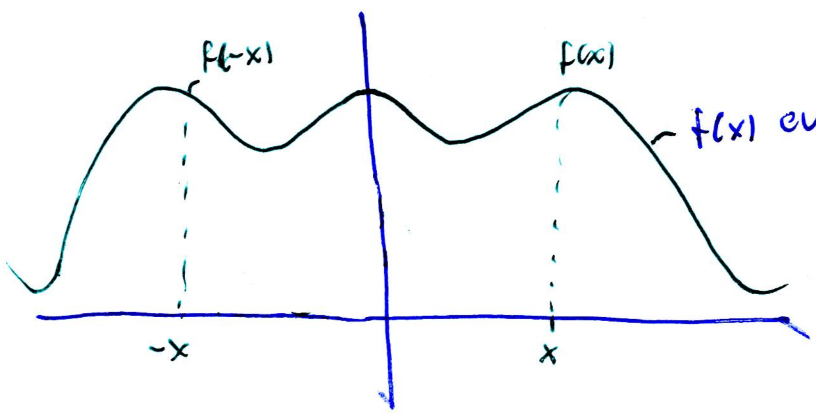
odd (odd) = odd   eg $\tan(x^{10})$

(4) $\int$ odd = even + c   eg $\int \sin x \, dx = -\cos x + c$

$\int$ even = odd. + c   eg $\int x^6 dx = \frac{x^7}{7} + c$

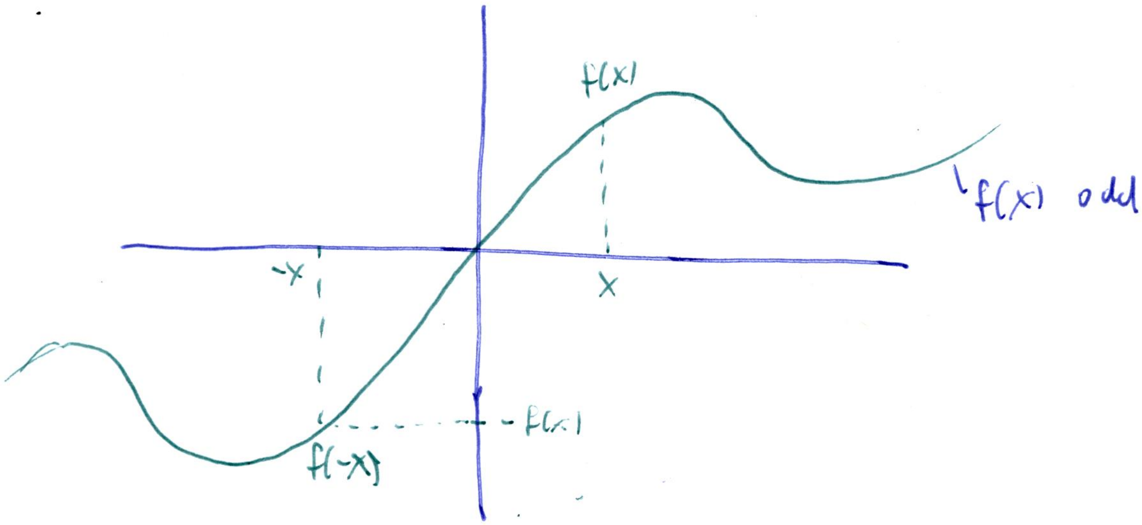
What do the graphs look like?

Recall the graph of $f(-x)$ is the graph of $f(x)$ flipped along the y-axis.
So if f is even, the $f(-x) = f(x)$ implies the graph doesn't change when flipped along y-axis.



$f(-x) = f(x)$

If f is odd then $f(-x) = -f(x)$ tells us the graph of f when flipped along the y-axis is the same as the graph of f flipped along the x-axis. eg



$$f(-x) = -f(x)$$

The reason why we are talking about them is so we can exploit their symetry.

If f is even the area under f from 0 to a is the same as the area under f from $-a$ to 0. So

$$f \text{ even} \Rightarrow \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

Similarly if f is odd, the area under f from 0 to a is same as the area under f from $-a$ to 0 but <u>opposite</u> sign. So,

$$f \text{ odd} \Rightarrow \int_{-a}^{a} f(x)\,dx = 0$$

eg $\int_{-50}^{50} (\arctan x^3)^5 \, dx$

$f(x) = (\arctan x^3)^5$, $f(-x) = (\arctan[(-x)^3])^5$

$= (\arctan(-x^3))^5$, since x^3 is odd

$= (-\arctan(x^3))^5$, since $\arctan x$ is odd

$= -[\arctan(x^3)]^5$, since x^5 is odd

$= -f(x)$

Thus f is odd and $\int_{-50}^{50} (\arctan x^3)^5 \, dx = 0$

eg $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x \, dx = 2 \int_0^{\frac{\pi}{4}} \cos x \, dx$

$= 2 \left[\sin x \right]_0^{\pi/4}$

$= 2 \sin \frac{\pi}{4}$

$= 2 \frac{1}{\sqrt{2}}$

$= \sqrt{2}$